Quantum Field Theory

Set 5

Exercise 1: Generator of rotations

Consider the Lagrangian of a massive vector field. Show that the Noether current associated to Lorentz transformation is

$$J^{\mu}_{\alpha\beta} = -F^{\mu\rho}\left(\eta_{\rho\alpha}A_\beta + x_\alpha \partial_\beta A_\rho\right) + F^{\mu\rho}\left(\eta_{\rho\beta}A_\alpha + x_\beta \partial_\alpha A_\rho\right) + \left(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha\right)\mathcal{L}.$$

Consider the component J^0_{ij} and thus find the Noether charge associated to rotations. Show that

$$J_k = \frac{1}{2}\epsilon_{ijk}\int J^0_{ij} d^3x = \epsilon_{ijk}\int d^3x \left(\Pi_i A_j - \Pi_m\, x_i \partial_j A_m\right).$$

The goal of this exercise is to compute the generator of rotations in terms of ladder operators.

Let us start by a not fully rigorous derivation, by guessing the structure of the result. We expect J_i to be at most quadratic in terms of ladder operators and in k_i, $\frac{\partial}{\partial k^j}$ (why?). Also it must be Hermitian and it must transform as a vector under rotation. This fixes the most general form of J_i to:

$$J_i = L_i + S_i,$$

$$L_k = i\epsilon_{ijk}B\int d\Omega_{\vec{k}}\, a_m(\vec{k})\left(k_i \frac{\partial}{\partial k^j}\right)a^\dagger_m(\vec{k}), \qquad S_k = i\epsilon_{ijk}A\int d\Omega_{\vec{k}}\, a_i(\vec{k})a^\dagger_j(\vec{k}).$$

As you can check, the orbital part $\vec{L}$ and the spin $\vec{S}$ commute between them: $[S_i, L_j] = 0$. Indeed they correspond to two distincts parts of the transformation of the field A_μ. Thus the only thing to do is to fix the coefficients A and B. Do this by requiring that the operator satisfies the expected $SO(3)$ commutation rules

$$[S_i, S_j] = i\epsilon_{ijk}S_k \qquad \textbf{and} \qquad [L_i, L_j] = i\epsilon_{ijk}L_k.$$

Explicit computation: Substitute the expansion of $\Pi_i(x)$ and $A_j(x)$ in terms of $a_i(k)$ and $a_i(k)^\dagger$ and compute the generator of rotations in terms of ladder operators.

Exercise 2: Rotations of a massive spin-1 state

In the last exercise you derived the spin operator. This is the only relevant part of the angular momentum for a particle in the rest frame. A generic massive single particle spin-1 state in the rest frame can be written as:

$$|\vec{\alpha}\rangle \equiv \alpha_i a^\dagger_i(\vec{0})\,|0\rangle = \vec{\alpha}\cdot\vec{a}^\dagger(\vec{0})\,|0\rangle\,,$$

where α_i are three coefficients. We want to consider the action of the spin operator on such a state. Then:

- Show that $a^\dagger_i$ obeys the right commutation rules of a vector with the spin:

$$[S_i, a^\dagger_j(\vec{0})] = i\epsilon_{ijk}a^\dagger_k(\vec{0}).$$

- Use the previous result to compute the action of S_3 on $|\vec{\alpha}\rangle$. Then diagonalize S_3 on the subspace spanned by the states $|\vec{\alpha}\rangle$, i.e. solve the eigenvalue equation:

$$S_3\,|\vec{\alpha}_a\rangle = \lambda_a\,|\vec{\alpha}_a\rangle \qquad \text{(no sum over } a\text{)}.$$

- Compute the action of $S^2 = S_iS_i$ on a state $|\vec{\alpha}\rangle$.

Exercise 3: Boost of a polarization vector

Consider a massive vector field of momentum $p^\mu = (E, 0, 0, p)$, with positive helicity, i.e. with $\varepsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0)$. Perform a boost along the y direction. Write the polarization vector after the boost and decompose it explicitly in the basis of polarization vectors with definite helicity.
Hint: after the boost, perform a rotation to align the momentum with the z axis.

Exercise 4: Parity transformation properties of a particle-antiparticle system

Consider a scalar particle and antiparticle pair in their center of mass frame. Assume their total angular momentum to be l. Hence this state can be written as

$$|\Phi_l\rangle = \int d\Omega_{\vec{p}}\ f_l(\vec{p}, -\vec{p}) a^\dagger(\vec{p}) b^\dagger(-\vec{p})|0\rangle.$$

Recalling the symmetry properties of a state with angular momentum l, i.e. $f_l(\vec{p}, -\vec{p}) = (-1)^l f_l(-\vec{p}, \vec{p})$, and the action of parity on scalars

$$P^\dagger a^\dagger(\vec{k})P = \eta_P a^\dagger(-\vec{k}), \qquad P^\dagger b^\dagger(\vec{k})P = \eta_P b^\dagger(-\vec{k}),$$

find the transformation properties of the state $|\Phi_l\rangle$ under P.
Consider now a generic state composed of a fermionic particle-antiparticle pair with angular momentum l and total spin S:

$$|\Psi_{l,S}\rangle = \sum_{r,t=1}^{2} \int d\Omega_{\vec{p}}\ f_l(\vec{p}, -\vec{p})\, \chi_S(r,t)\, \tilde{d}^\dagger(\vec{p}, r) b^\dagger(-\vec{p}, t)|0\rangle,$$

where $\tilde{d}^\dagger(\vec{p}, r) \equiv d^\dagger(\vec{p}, r')\epsilon^{rr'}$, $r = 1, 2$ creates an antiparticle with spin $+1/2, -1/2$ respectively (not $-1/2, 1/2$ as it would be for $d^\dagger(\vec{p}, r)$). The action of parity is defined as

$$P^\dagger b^\dagger(\vec{k}, r)P = \eta_P b^\dagger(-\vec{k}, r), \qquad P^\dagger \tilde{d}^\dagger(\vec{k}, t)P = -\eta_P \tilde{d}^\dagger(-\vec{k}, t),$$

and the wave functions satisfy

$$f_l(\vec{p}, -\vec{p}) = (-1)^l f_l(-\vec{p}, \vec{p}), \qquad \chi_S(r,t) = (-1)^{S+1}\chi_S(t, r).$$

Find the transformation properties of the state $|\Psi_{l,S}\rangle$ under P.